Visibility Determination

Nick Elias
OAM Solutions, LLC
2014 July 21

1.0 Introduction

In this document, I rederive the formulae for determining squared visibilities from NPOI fringe data.

2.0 NPOI Response and Measurement Equations

The NPOI response for one wavelength of an output beam is

$$I(x) = \sum_{i=1}^{M} I_i + 2\sum_{l=1}^{M}\sum_{\substack{m=1\\m\neq l}}^{M}\sqrt{I_l I_m}\,|\gamma_{lm}|\cos\left(2\pi\frac{k\,x}{\lambda}+\psi_{lm}\right),$$

where M is the number of siderostats, x is the delay, λ is the wavelength, I_i is the average photon rate per delay for each siderostat (input beam) *i*, $|\gamma_{lm}|$ is the correlation amplitude for baseline *m*-*n*, ψ_{lm} is the correlation phase, and k is the modulation index ($k \to k_{lm}$ for baseline *m*-*n*). The correlation quantities include the effects of source structure (scalar and polarized), atmospheric turbulence, and instrumental imperfections (scalar and polarized) which much eventually be calibrated. This equation corresponds to Equation 1 in Hummel *et. al.* (2003).

One pass across a fringe is called a frame. The actual measurements across a frame are represented by a binned version of the previous equation

$$b_j = \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} dx\, I(x) = \overline{N} + 2\sum_{l=1}^{M}\sum_{\substack{m=1\\m\neq l}}^{M}\sqrt{\overline{N}_l\overline{N}_m}\,|\gamma_{lm}|\,Sa\left(\frac{\pi\,k}{n}\right)\cos\left(2\pi\frac{k\,j}{n}+\psi_{lm}\right),$$

where $n = 64$ is the number of bins, $0 \le j \le n-1$ is the bin number, Δx is the bin width, $\overline{N}_i$ is the average number of photons per bin for siderostat *i*, $\overline{N} = \sum_{i=1}^{M}\overline{N}_i = \sum_{i=1}^{M}\left(\Delta x\, I_i\right)$ is the average number of photons per bin, and $Sa(x) = \sin(x)/x$ is the sampling function. The sampling function becomes independent of k and approaches unity as the number of bins approaches infinity. Note that NPOI creates phase bins, i.e., the same number of bins span a fringe independent of wavelength, or the bin width is effectively a function of (2003); the latter is not quite correct because it has a factor of 0.5 multiplying to correlation sum instead of 2.0.

wavelength $\Delta x \to \Delta x(\lambda)$. This equation corresponds to Equation 2 in Hummel *et al.*

3.0 X, Y, and N

The real part of the visibility is

$$X = \sum_{j=0}^{n-1} b_j \cos\left(2\pi \frac{k\,j}{n}\right) = N\,\delta_{k0} + \sum_{l=1}^{M} \sum_{\substack{m=1 \\ m\neq l}}^{M} \sqrt{N_l N_m}\,|\gamma_{lm}|\cos\psi_{lm}\; Sa\left(\frac{\pi\,k}{n}\right),$$

where $N_i = n\,\overline{N}_i$ is the total number of photons per frame for siderostat *i*, and $N = n\,\overline{N}$ is the total number of photons per frame. The $k = 0$ term yields N. Similarly, the imaginary part of the visibility is

$$Y = \sum_{j=0}^{n-1} b_j \sin\left(2\pi \frac{k\,j}{n}\right) = \sum_{l=1}^{M} \sum_{\substack{m=1 \\ m\neq l}}^{M} \sqrt{N_l N_m}\,|\gamma_{lm}|\sin\psi_{lm}\; Sa\left(\frac{\pi\,k}{n}\right).$$

Assuming no complications from rapid turbulence, etc., I assume that I can obtain the real and imaginary parts of the desired baseline *l-m* at an integral *k*, or

$$X_{lm} = \sqrt{N_l N_m}\,|\gamma_{lm}|\cos\psi_{lm}\; Sa\left(\frac{\pi\,k_{lm}}{n}\right)$$

and

$$Y_{lm} = \sqrt{N_l N_m}\,|\gamma_{lm}|\sin\psi_{lm}\; Sa\left(\frac{\pi\,k_{lm}}{n}\right).$$

4.0 Squared Visibility

The observed squared visibility for an output beam with two telescopes is

$$|V_{lm}|^2 = 4\frac{X_{lm}^2 + Y_{lm}^2}{(N_l + N_m)^2} = 4\left[\frac{\alpha_{ml}}{(1+\alpha_{ml})^2}\right] Sa^2\left(\frac{\pi\,k_{lm}}{n}\right)|\gamma_{lm}|^2,$$

where $\alpha_{ml} = N_m / N_l$ is the photon number ratio from siderostats *m* and *l*. It depends on the photometric behavior of the instrument (scalar and vector) and source (vector). The quantity in square brackets is 1/4 when the photon number ratios are unity. For NPOI, the α_{ml} should not deviate too far from unity, so using $\alpha_{ml} = 1$ will provide reasonable first-look estimates of the squared visibilities.

The quantity of interest is $|\gamma_{lm}|^2$, which must eventually be calibrated. The factor containing the squared sampling function is known. The factor involving the photon number ratio, on the other hand, is not known because the number of photons from the individual siderostats are not known. If the instrument is stable, it is not necessary to know this factor exactly because calibration will remove it (Section 6). Note that the ratios can be determined by dedicated photometric calibration.

For output beams with M telescopes, $N_l + N_m$ can be replaced by the sum of photon numbers for the M telescopes, or $N = N_l + N_m + N_p + N_q + \ldots$. The general squared visibility equation then becomes

$$\left|V_{lm}\right|^2 = M^2 \frac{X_{lm}^2 + Y_{lm}^2}{\left(N_l + N_m + N_p + N_q + \cdots\right)^2} = M^2 \left[\frac{\alpha_{ml}}{\left(1 + \alpha_{ml} + \alpha_{pl} + \alpha_{ql} + \cdots\right)^2}\right] Sa^2\left(\frac{\pi k_{lm}}{n}\right) \left|\gamma_{lm}\right|^2 .$$

The quantity in square brackets is $1/M^2$ when the photon number ratios are unity.

5.0 Bias Removal

The bias of $X_{lm}^2 + Y_{lm}^2$ is $\Sigma^2 = \sum_{j=0}^{n-1} \sigma_j^2$, where σ_j^2 is the estimated variance of bin *j*. For Poisson noise, $\Sigma^2 = \sum_{j=0}^{n-1} b_j = N$. For non-Poisson noise processes, $\sigma_j^2 = \sigma_0 + \sigma_1 b_j$, which means that $\Sigma^2 = n\sigma_0 + N\sigma_1$ (for Poisson noise $\sigma_0 = 0$ and $\sigma_1 = 1$). The σ_0 (non-zero due to read noise) and σ_1 (non-unity due to APD after pulsing) parameters can be fit directly from the data for each output beam. APDs do not exhibit read noise, although in principle it is prudent to include σ_0 in the fit anyway. Also, it is also possible to include higher-order terms such as $\sigma_2 b_j^2$, but they ultimately depend on squared visibilities in addition to photon numbers. I will discuss this topic in a future version of this paper.

Removing the Poisson noise bias is not sufficiently accurate unless the photon number is relatively large. Hummel *et al.* (2003) performed a fit of

$$\log_{10}\left|V\right|^2 = K \log_{10} N + C$$

for off-fringe (calibrator and target) scans. The fit parameters change from night to night, but nominal values are $K = -1.63$ and $C = \log_{10} 7.7$.

Alternatively, I propose fitting for $n\sigma_0$ and σ_1 via a least-squares (it is not necessary to separate n and σ_0). The squared visibility for off-fringe scans is

$$\left|V_{off}\right|^2 = M^2 \frac{n\sigma_0 + \sigma_1 N}{N^2} = M^2 n\sigma_0 \frac{1}{N^2} + M^2 \sigma_1 \frac{1}{N},$$

and the fit parameters are $M^2 n\sigma_0$ and $M^2\sigma_1$. Deviations from pure Poisson noise are large for small N, so fitting versus $1/N$ and $1/N^2$ could be problematic. Therefore, I multiply both sides of this equation by N^2/M^2 to obtain

$$2\frac{N}{M} SNR_{bias} = \left[\frac{N}{M}\right]^2 \left|V_{off}\right|^2 = n\sigma_0 + \sigma_1 N,$$

where

$$SNR_{bias} = \frac{1}{2}\frac{N}{M}\left|V_{off}\right|^2$$

is the fringe-amplitude signal-to-noise ratio for an output beam normalized to two telescopes. This equation is now just a very simple (weighted) linear fit, where the dependent variable is proportional to the photon number times the signal-to-noise, the independent variable is the photon number, and the fit parameters are an offset and slope.

Dark current is the only source of bias for the denominator. For the purposes of this document, I assume that it is negligible.

6.0 Calibration

Elementary calibration involves observations of an unresolved reference star or a reference star of known diameter close (in time and great-circle angle) to the target star,

$$\frac{|V_{lm}|^2}{|V_{lm}^{ref}|^2} = \frac{|\gamma_{lm}|^2}{|\gamma_{lm}^{ref}|^2} \rightarrow |\gamma_{lm}|^2 = |\gamma_{lm}^{ref}|^2 \frac{|V_{lm}|^2}{|V_{lm}^{ref}|^2}.$$

The photon number ratios and the sampling functions cancel. If the star is unresolved, $|\gamma_{lm}^{ref}|^2 \approx 1$, otherwise $|\gamma_{lm}^{ref}|^2 \approx 2\frac{J_1(\pi B_{lm}\theta/\lambda)}{\pi B_{lm}\theta/\lambda}$, where $J_1(x)$ is the first-order Bessel function of the first kind, B_{lm} is the length of baseline *l-m*, and θ is the uniform disk diameter (rad). The errors scale accordingly.

There is another way to calibrate squared visibilities. Calculate the system visibility for all on-fringe calibrator scans (and channels)

$$|V_{lm,i}^{sys}|^2 = \frac{|V_{lm,i}|^2}{|\gamma_{lm,i}^{ref}|^2},$$

where i is the on-fringe calibrator scan number, $|\gamma_{lm,i}^{ref}|^2 \approx 2\frac{J_1(\pi B_{lm,i}\theta/\lambda)}{\pi B_{lm,i}\theta/\lambda}$, and $B_{lm,i}$ is the length of baseline *l-m* for the i^{th} on-fringe calibrator scan number. Once the system visibilities of the calibrator on-fringe scans have been determined, they can be used to create a weighted estimate of the system visibilities of the target on-fringe scans

$$|V_{lm,j}^{sys}|^2 = \frac{\sum_i w_{ji}|V_{lm,i}^{sys}|^2}{\sum_i w_{ji}},$$

where j is the on-fringe target scan number, and the w_{ji} are the weights. I consider three possible weighting schemes

- Time difference: $w_{ji} = \frac{1}{\sigma_i^2\left[1+(t_j - t_i)^2/T^2\right]}$, where σ_i^2 is the variance of $|V_{lm,i}^{sys}|^2$, t_j is the time corresponding to the j^{th} on-fringe target scan time, t_i is the time corresponding to the i^{th} on-fringe calibrator scan time, and T is a scale time (nominally 0.5-1.0 hours).
- Angle difference: $w_{ji} = \frac{1}{\sigma_i^2\left[1+\Delta A_{ji}/A\right]}$, where ΔA_{ji} is the great circle distance between the target at the j^{th} on-fringe scan and the calibrator at the i^{th} on-fringe scan, and A is a scale great circle distance (nominally 10-20 degrees).

- Time and angle difference: $w_{ji} = \frac{1}{\sigma_i^2} \frac{1}{\left[1 + \left(t_j - t_i\right)^2 / T^2\right]} \frac{1}{\left[1 + \Delta A_{ji} / A\right]}$.

Once the system visibilities of the target stars have been estimated, they can be used to produce completely calibrated visibilities of the target stars

$$\left|\gamma_{lm,j}\right|^2 = \frac{\left|V_{lm,j}\right|^2}{\left|V_{lm,j}^{sys}\right|^2}.$$

The errors scale accordingly. If the calibrator diameters are well known, their system visibilities should depend only on atmospheric and instrumental effects. If the calibrator diameters are not well known, however, it is possible to associate each target star to a calibrator star and determine each target system visibility with only the associated calibrator system visibilities. The calibrated visibilities for a single target star will be internally consistent, but there is no guarantee of consistency when calibrated visibilities of different target stars are compared. Plus, the number of calibrator system visibilities grouped with each target visibility is much less than the total number of calibrator visibilities, which means that systematic errors could be larger.